

Hamilton-Jacobi quantization of singular Lagrangians with linear velocities

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ABSTRACT

In this paper, constrained Hamiltonian systems with linear velocities are investigated by using the Hamilton-Jacobi method. We shall consider the integrability conditions on the equations of motion and the action function as well in order to obtain the path integral quantization of singular Lagrangians with linear velocities.

1 Introduction

The study of singular Lagrangian with linear velocity has been dealt within the last 50 years by Dirac's Hamiltonian formalism [1, 2]. In this formalism Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: so-called first-class and second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first-class are by definition second-class. Also in his method, the Poisson brackets in a second class constraints systems are converted into Dirac brackets to attain self-consistency. However, wherever we adopt the Dirac method, we frequently meet the problem of the operating ordering ambiguity. Besides, the presence of first class constraints in such theories requires care when applying Dirac's method, since the first class constraints are the generators of gauge transformations which lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional arbitrariness, one has to impose external gauge fixing constraint for each first class constraint which is not always an easy task.

Recently, Güler [3-6] have proposed an alternating approach to constrained systems that avoids the separations of constraints into first and second class and the use of weak and strong equations. This new method of analysis has been successfully used by many authors [7-9] and is by now a standard technique to deal with constrained system. Besides, the canonical path integral method based on the Hamilton-Jacobi method have been initiated in [10] to obtain that path integral quantization as an integration over the canonical phase space coordinate without any need to use any gauge fixing conditions [7] as well as, no need to enlarge the initial phase-space by introducing unphysical auxiliary fields [11,12].

Some authors [9, 13-16] have investigated singular Lagrangians with linear velocities using Dirac's procedure. For example, using the Lagrangian given in reference [15], there are different approaches which give different results. Besides, in Ref. [13] the authors have investigated singular Lagrangian with linear velocities without considering the integrability condition on the action function. On the other hand in reference [9], the authors have investigated singular Lagrangian with linear velocities by using the Hamilton-Jacobi method and obtained the integrable action directly without considering the total variation of constraints.

In this paper, we shall consider integrability condition on the equations of motions and the action function as well in order to obtain the path integral quantization of singular Lagrangian with linear velocities.

2 The Hamilton-Jacobi method

In this section, we shall briefly review the Hamiltonian formulation of constrained systems [3-6]. The starting point of this method is to consider the Lagrangian $L = L(q_i, \dot{q}_i, t)$, $i = 1, \dots, n$, with the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \dots, n, \quad (1)$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \mu = n - r + 1, \dots, n, \quad (3)$$

where q_i are divided into two sets, q_a and x_μ . Since the rank of the Hessian matrix is $(n - r)$, one solve Eq. (10) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \quad (4)$$

Substituting Eq. (12) into Eq. (11), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \quad (5)$$

The canonical Hamiltonian H_0 reads

$$H_0 = p_a \dot{q}_a + p_\mu \dot{x}_\mu|_{p_\nu = -H_\nu} - L(t, q_i, \dot{x}_\nu, \dot{q}_a), \quad \mu, \nu = n - r + 1, \dots, n. \quad (6)$$

The set of Hamilton-Jacobi partial differential equations [HJPDE] is expressed as [3-6]

$$H'_\alpha \left(x_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial x_\alpha} \right) = 0, \quad \alpha, \beta = 0, n - r + 1, \dots, n, \quad (7)$$

where

$$H'_0 = p_0 + H_0, \quad (8)$$

$$H'_\mu = p_\mu + H_\mu, \quad (9)$$

we define $p_\beta = \partial S[q_a; x_\alpha]/\partial x_\beta$ and $p_a = \partial S[q_a; x_\alpha]/\partial q_a$ with $x_0 = t$ and S being the action. The equations of motion are obtained as total differential equations in many variables as follows [3-6]:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dx_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dx_\alpha, \quad (10)$$

$$dp_\beta = -\frac{\partial H'_\alpha}{\partial x_\beta} dx_\alpha, \quad (11)$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dx_\alpha, \quad (12)$$

where $z = S(x_\alpha; q_a)$. The analysis of a constrained system is reduced to solve equations (10) with constraints $H'_\alpha = 0$. Variation of constraints (7) considering equations (10) may vanish identically or give rise to new constraints. In the case of new constraints we should consider their variations also. Repeating this procedure, one may obtain a set of constraints such that all the variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the solution of a singular system. In fact, in references [7], the integrability conditions for equations (10, 11) are discussed without considering the integrability conditions of the action function.

3 Completely and Partially Integrable Systems

As was clarified, that the equations (10-12) are obtained as total differential equations in many variables, which require the investigation of integrability conditions. To achieve this goal we define the linear operators X_α which corresponds to total differential equations (6,7) as

$$\begin{aligned} X_\alpha f(t_\beta, q_a, p_a, z) &= \frac{\partial f}{\partial t_\alpha} + \frac{\partial H'_\alpha}{\partial p_a} \frac{\partial f}{\partial q_a} - \frac{\partial H'_\alpha}{\partial q_a} \frac{\partial f}{\partial p_a} \\ &\quad + \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) \frac{\partial f}{\partial z}, \\ &= [H'_\alpha, f] - \frac{\partial f}{\partial z} H'_\alpha, \\ \alpha, \beta &= 0, n-r+1, \dots, n, a = 1, \dots, n-r, \end{aligned} \quad (13)$$

where the commutator $[,]$ is the square bracket (for details, see the appendix).

lemma. A system of total differential equations (10-12) is integrable if and only if

$$\{H'_\alpha, H'_\beta\} = 0, \quad \forall \alpha, \beta, \quad (14)$$

where the commutator $\{, \}$ is the Poisson bracket (for details see references [8, 11, 12]).

Equations (14) are the necessary and sufficient conditions that the system (10-12) of total differential equations be completely integrable and we call this system as *Completely Integrable Model* . However, equations (10,11) form here by themselves a completely integrable system of total differential equations. If these are integrated, then only simple quadrature has to be carried out in order to obtain the action [7].

On the other hand, we must emphasis that the total differential equations (10,11) can be very well be completely integrable without (14) holding and therefore without the total system (10-12) being integrable and we call this system as *Partially Integrable Model*. In fact, if $\{H'_\beta, H'_\alpha\} = F_m(t, t_\mu)$, where F_m are functions of t_α and m is integer, then the total differential equations (10, 11), will be integrable [8, 11, 12].

If the set of equations (10-12) is integrable, then one can obtain the canonical action function (12) in terms of the canonical coordinates. In this case, the path integral representation may be written as [10]

$$\begin{aligned} \Psi(q'_a, t'_\alpha; q_a, t_\alpha) &= \int_{q_a}^{q'_a} Dq^a Dp^a \times \\ &\exp i \left\{ \int_{t_\alpha}^{t'_\alpha} \left[-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right] dt_\alpha \right\}, \\ a &= 1, \dots, n-r, \quad \alpha = 0, n-r+1, \dots, n. \end{aligned} \quad (15)$$

4 The Model

In this section we would like to investigate singular with linear velocities by using the Hamilton-Jacobi method [3-6, 8, 12], in order to obtain the path integral quantization for these systems. Let us consider the following linear Lagrangian [9, 13]

$$L = a_i(q_j) \dot{q}_i - V(q_j), \quad (16)$$

where $a_i(q_j)$ and $V(q_j)$ are continues functions of q_j .

The generalized canonical momentum corresponding to this Lagrangian are given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = a_i(q) = -H_i. \quad (17)$$

The primary constraints are

$$H'_i = p_i - a_i. \quad (18)$$

The canonical Hamiltonian H_0 is given by:

$$H_0 = p_i \dot{q}_i - L = V(q_j). \quad (19)$$

The corresponding HJPDEs are

$$H'_0 = p_0 + H_0 = p_0 + V(q_j) = 0, \quad (20)$$

$$H'_i = p_i + H_i = p_i - a_i(q_j) = 0. \quad (21)$$

The equations of motion are obtained as total differential equations follows:

$$dq_i = \frac{\partial H'_0}{\partial p_i} dt + \frac{\partial H'_j}{\partial p_i} dq_j = dq_i, \quad (22)$$

$$dp_i = -\frac{\partial H'_0}{\partial q_i} dt - \frac{\partial H'_j}{\partial q_i} dq_j = -\frac{\partial V(q)}{\partial q_i} dt + \frac{\partial a_j(q)}{\partial q_i} dq_j. \quad (23)$$

To check whether the set of equations (22) and (23) are integrable or not, let us consider the total variation of (21). In fact

$$\begin{aligned} dH'_i &= dp_i - da_i(q) = 0 \\ &= -\frac{\partial V(q)}{\partial q_i} dt + \frac{\partial a_j(q)}{\partial q_i} dq_j - da_i(q) \end{aligned} \quad (24)$$

So, we have

$$\frac{\partial a_i(q)}{\partial q_j(q)} dq_j - \frac{\partial a_j(q)}{\partial q_i} dq_j = -\frac{\partial V(q)}{\partial q_i} dt, \quad (25)$$

or

$$\dot{q}_j = -f_{ij}^{-1} \frac{\partial V(q)}{\partial q_i}, \quad (26)$$

where the anti-symmetric matrix f_{ij} is given by

$$f_{ij} = \frac{\partial a_i(q)}{\partial q_j(q)} - \frac{\partial a_j(q)}{\partial q_i}. \quad (27)$$

Making use of (12), (20) and (21), we can write the canonical action integral as

$$S = \int a_i dq_i - \int V(q_j) dt. \quad (28)$$

In fact,

$$\int d(a_i q_i) = a_i q_i = \int a_i dq_i + \int q_i da_i. \quad (29)$$

So

$$S = \frac{1}{2}a_i q_i - \frac{1}{2} \int [q_j da_i - a_i dq_i + 2V(q)dt]. \quad (30)$$

Making use of (24) and (29), the action function becomes

$$S = \frac{1}{2}a_i q_i - \frac{1}{2} \int \left[\left(2V - q_j \frac{\partial V}{\partial q_i} \right) dt + \left(\frac{\partial a_j}{\partial q_i} q_j - a_i \right) dq_i \right]. \quad (31)$$

Now, assuming that the functions $a_i(q_j)$ and $V(q_j)$ satisfy the following conditions

$$2V = \frac{\partial V}{\partial q_i} q_j, \quad a_i = \frac{\partial a_j}{\partial q_i} q_j,$$

we obtain the integrable action as follows:

$$S = \frac{1}{2}a_i q_i + c, \quad (32)$$

where c is some constant.

To obtain the path integral quantization for the singular Lagrangian (16), we have three different cases,

Case 1: If the inverse of the matrix f_{ij} exists, then we can solve all the dynamics q_i . In this case the path integral Ψ is given by

$$\Psi = \int \prod_{i=1}^n dq_i e^{i(\frac{1}{2}a_i q_i)}. \quad (33)$$

Case 2: If the rank of the matrix f_{ij} is $n-R$, then we can solve the dynamics q_l in terms of independent parameters (t, q_α) , $\alpha = 1, 2, \dots, R$. In this case the path integral Ψ is calculated as

$$\Psi = \int \prod_{l=1}^{n-R} dq_l e^{i(\frac{1}{2}a_i q_i)}, \quad i = 1, \dots, n. \quad (34)$$

Case 3: If $a_i(q_j)$ are constants, then the path integral Ψ is given by

$$\Psi = e^{i \int_{q'}^{q''} (a_i dq_i)} = e^{ia_i(q_i'' - q_i')}, \quad (35)$$

which satisfies the "Wheeler-DeWitt" equation

$$(p_i - a_i)\Psi = 0. \quad (36)$$

This result coincide with the results obtained in reference [17] by using the naively applied Batalin, Fradkin, Vilkovisky procedure [18].

5 Examples

As a first example, we consider the following linear (singular) Lagrangian [9]

$$L = q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_1 - q_1\dot{q}_3 - V, \quad (37)$$

where the potential V is given by

$$V = 2q_1q_3 - \frac{1}{2}q_3^2. \quad (38)$$

The functions $a_i (i = 1, 2, 3)$ are

$$a_1 = q_1 + q_3, \quad a_2 = q_2, \quad a_3 = -q_1. \quad (39)$$

Using (17), the generalized momenta corresponding to this Lagrangian are:

$$\begin{aligned} p_1 &= a_1 = q_1 + q_3 = -H_1, \\ p_2 &= a_2 = q_2 = -H_2, \\ p_3 &= a_3 = -q_1 = -H_3. \end{aligned}$$

By (18) the primary constraints are given as

$$H'_1 = p_1 - q_1 - q_3, \quad H'_2 = p_2 - q_2, \quad H'_3 = p_3 + q_1. \quad (40)$$

Equation (19) gives the canonical Hamiltonian H_0 as

$$H_0 = V(q) = 2q_1q_3 - \frac{1}{2}q_3^2. \quad (41)$$

Now using (23), the equations of motion read as

$$dp_1 = -2q_3dt + dq_1 - dq_3, \quad (42)$$

$$dp_2 = dq_2, \quad (43)$$

$$dp_3 = (-2q_1 + q_3)dt + dq_1. \quad (44)$$

The matrix f_{ij} defined in (27) is given by

$$f_{ij} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}. \quad (45)$$

Making use of (24), we can obtained the equation of motion for q_1, q_2 and q_3 respectively as

$$dq_3 + q_3dt = 0, \quad 0 = 0, \quad -2dq_1 + (2q_1 - q_3)dt = 0. \quad (46)$$

The integrable action function is calculated as

$$S = \frac{1}{2} (q_1^2 + q_2^2) + c. \quad (47)$$

Making use of equation (34) and equation (47), the path integral for the model is given by

$$\Psi = \int dq_1 dq_3 e^{i\frac{1}{2}(q_1^2 + q_3^2)}. \quad (48)$$

As a second example, let us consider the following linear (singular) Lagrangian [9]

$$L = (q_2 + q_3) \dot{q}_1 + q_4 \dot{q}_3 - V(q), \quad (49)$$

where the potential $V(q)$ is given by

$$V(q) = -\frac{1}{2} (q_4^2 - 2q_2q_3 - q_3^2). \quad (50)$$

The functions $a_i (i = 1, 2, 3, 4)$ are

$$a_1 = q_2 + q_3, \quad a_2 = 0, \quad a_3 = q_4, \quad a_4 = 0. \quad (51)$$

Using (17), the generalized momenta corresponding to this Lagrangian are:

$$\begin{aligned} p_1 &= a_1(q) = q_2 + q_3 = -H_1, \\ p_2 &= a_2 = 0 = -H_2, \\ p_3 &= a_3 = q_4 = -H_3, \\ p_4 &= a_4 = 0 = -H_4. \end{aligned}$$

The primary constraints are given as

$$H'_1 = p_1 - q_2 - q_3, \quad H'_2 = p_2, \quad H'_3 = p_3 + q_4, \quad H'_4 = p_4. \quad (52)$$

Equation (19) gives the canonical Hamiltonian H_0 as

$$H_0 = V(q) = -\frac{1}{2} (q_4^2 - 2q_2q_3 - q_3^2). \quad (53)$$

Now making use of (23), the equations of motion read as

$$dp_1 = 0, \quad (54)$$

$$dp_2 = -q_3 dt + dq_1, \quad (55)$$

$$dp_3 = -(q_2 + q_3) dt + dq_1, \quad (56)$$

$$dp_4 = q_4 dt + dq_3. \quad (57)$$

The matrix f_{ij} defined in (27) is given by

$$f_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (58)$$

The S -action function is calculated as

$$S = \frac{1}{2} [(q_2 + q_3)q_1 + q_4q_3] + c. \quad (59)$$

Making use of (33) and (59), the path integral for the model is calculated as

$$\Psi = \int dq_1 dq_2 dq_3 dq_4 e^{i\frac{1}{2}[(q_2+q_3)q_1+q_4q_3]}. \quad (60)$$

An important point to be specified here is that, in the Jackiw's method treatment of the above model [19], the path integral is obtained by introducing δ function in the measure of the integral, while in the canonical path integral method [10], the path integral quantization is obtained directly as an integration over the canonical variables q_1, q_2, q_3, q_4 without any need to use these δ functions.

6 Conclusion

In this work we have investigated constrained systems using the Hamilton-Jacobi method for Lagrangians with linear velocities. The equations of motion are obtained from the integrability conditions and the number of independent parameters (*multi-times*) are determined from the rank of matrix f_{ij} . Besides the integrable action is obtained from the integrability conditions, which leads us to obtain the path integral quantization for the singular Lagrangians with linear velocities directly as an integration over the independent dynamical variables without any need to use δ functions as given in the Faddeev, Jackiw method [19].

7 Square brackets and Poisson brackets

In this appendix we shall give a brief review on two kinds of commutators: the square and the Poisson brackets.

The square bracket is defined as

$$[F, G]_{q_i, p_i, z} = \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial p_i} (p_i \frac{\partial G}{\partial z}) - \frac{\partial G}{\partial p_i} (p_i \frac{\partial F}{\partial z}). \quad (61)$$

The Poisson bracket is defined as

$$\{f, g\}_{q_i, p_i} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i}. \quad (62)$$

According to above definitions, the following relation holds

$$[H'_\alpha, H'_\beta] = \{H'_\alpha, H'_\beta\}. \quad (63)$$

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